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# Instantons, Integrability and Discrete Light-Cone Quantisation

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## Abstract

We study supersymmetric quantum mechanics on the moduli space of Yang-Mills instantons on  $\mathbb{R}^2 \times T^2$  and its application to the discrete light-cone quantisation (DLCQ) of  $\mathcal{N} = 4$  SUSY Yang-Mills. In the presence of a target space magnetic field, the model has a discrete spectrum with the wavefunctions of generic energy eigenstates supported away from the singular points of the moduli space. The corresponding Hamiltonian is part of an  $\mathfrak{osp}(1, 1|4)$  superalgebra which enlarges to  $\mathfrak{su}(1, 1|4)$  superconformal invariance in the sector corresponding to the  $\mathcal{N} = 4$  theory. The Hamiltonian is isospectral to the light-cone dilatation operator of the  $\mathcal{N} = 4$  theory in this sector. The model also has an interesting scaling limit where it becomes integrable. We determine the semiclassical spectrum in this limit. We discuss a possible approach to constructing the dilatation operator of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory in DLCQ.

# 1 Introduction

The developments of the last few years strongly suggest the existence of hidden symmetries in non-abelian gauge theory which are not manifest in the standard Lagrangian formulation of the theory. This is particularly apparent in the case of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory where remarkable simplifications occur in perturbative computations of operator dimensions, scattering amplitudes and other observables. In the planar limit, there is overwhelming evidence that the theory is exactly integrable and the assumption of integrability leads to an exact solution for the anomalous dimensions of all local operators in this limit [1]. Despite this, the origin of integrability and of the corresponding symmetries in gauge theory remains obscure. In particular, the spin chain Hamiltonian is corrected by longer range interactions order by order in perturbation theory and its persistent integrability at each new order seems miraculous. The existing bootstrap solution of the spectral problem provides detailed information about the eigenvalues of the planar dilatation operator but not the underlying Hamiltonian or the corresponding wave functions.

For the reasons described above, it is natural to look for a formulation of the  $\mathcal{N} = 4$  theory in which some or all of the hidden symmetries become manifest<sup>1</sup>. With this goal in mind, we will study the  $\mathcal{N} = 4$  theory with gauge group  $SU(N)$  in Discrete Light-Cone Quantisation (DLCQ) following the proposal of [3, 4]. In this formalism, one studies the theory compactified on a light-like circle of fixed radius  $R_-$ , so that the corresponding null momentum is quantised in integer units:  $p_+ = K/R_-$ ,  $K \in \mathbb{N}$ . As usual, restricting the theory to a sector with  $K$  units of null momentum reduces a field theory to a finite dimensional quantum mechanical model with a number of degrees of freedom which grows linearly with  $K$ . An obvious hope is that the finite-dimensional model might itself be integrable or, more realistically, might become so for large  $N$ . In this paper, we will give a precise formulation

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<sup>1</sup>One possible candidate is provided by the worldsheet theory of the dual string which is classically integrable. However, the first-principles quantisation of the worldsheet  $\sigma$ -model for the Green-Schwarz string in  $AdS_5 \times S^5$  is a hard problem which remains unsolved. The main obstacle is in finding a discretised version of the worldsheet theory in which integrability is preserved. For recent progress in this direction, see however [2].

of the relevant quantum mechanical model and show that it is indeed at least semiclassically integrable in a certain limit (even for finite  $N$ ). The relation to the integrability of the planar  $\mathcal{N} = 4$  theory remains to be understood. Along the way we will also perform some basic checks on the proposed DLCQ description.

As we review below, the DLCQ description of the  $\mathcal{N} = 4$  theory with gauge group  $SU(N)$  in the sector with  $K$  units of null momentum is given by supersymmetric quantum mechanics on the moduli space of  $K$  instantons of an auxiliary  $SU(N)$  gauge theory formulated on  $\mathbb{R}^2 \times T^2$ . More precisely, we should take a limit where the area of the torus goes to zero. The model also has additional parameters which correspond to light-like Wilson lines breaking the gauge group of the DLCQ theory down to its Cartan subgroup. This proposal of [3, 4] has its origin in the realisation of  $\mathcal{N} = 4$  SUSY Yang-Mills as an IR fixed point arising when the six-dimensional  $(2, 0)$  theory is compactified to four dimensions on a torus [5]. An attractive feature of this approach is that dependence on the gauge theory coupling is encoded geometrically in the complex structure of the torus.

In this paper we first establish some basic results about the proposed DLCQ description. The moduli space of  $K$   $SU(N)$  instantons on  $\mathbb{R}^2 \times T^2$  is a hyper-Kähler manifold<sup>2</sup> of real dimension  $4r = 4(KN - N + 1)$  which we will denote as  $\mathcal{M}$ . Although the full hyper-Kähler metric on  $\mathcal{M}$  is complicated, it approaches a simple analytic form known as the *semi-flat metric* in the limit relevant for describing the  $\mathcal{N} = 4$  theory. Building on the results of our previous paper [6], we construct the relevant quantum mechanical  $\sigma$ -model with target space  $\mathcal{M}$  equipped with the semi-flat metric. An important qualitative feature of the moduli space is that each  $SU(N)$  instanton splits into  $N$  constituents or “partons” moving on  $\mathbb{R}^2 \times T^2$ . The model has a weak coupling limit where the semi-flat metric becomes flat and the system consists of  $KN$  free partons<sup>3</sup>. Away from this limit, the partons interact with each other through the curvature of the moduli space metric. Despite this, the semi-flat  $\sigma$ -model has  $2r$  conserved charges<sup>4</sup>  $\{Q_{eI}, Q_m^I\}$ ,  $I = 1, 2, \dots, r$ , corresponding to the momenta of *individual*

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<sup>2</sup>The manifold has singularities which we will discuss below.

<sup>3</sup>More precisely there are  $K$  identical partons in each of  $N$  species with  $N - 1$  linear constraints on the relative positions of the center of mass of each species.

<sup>4</sup>As we explain below,  $\mathcal{M}$  can be understood as a torus fibration over a special Kähler base. Special Kähler geometry provides a symplectic pairing between “electric” and “magnetic” cycles of the fibre denoted by the labels  $e$  and  $m$  respectively.

partons in the two compact directions. Although this is only half the number of conserved charges needed to make the  $\sigma$ -model integrable, we will describe an interesting limit where the system nevertheless becomes integrable.

In the context of DLCQ,  $Q_{eI}$  and  $Q_m^I$  correspond to momenta carried by the tower of Kaluza-Klein states in the compactification of the  $(2,0)$ -theory on  $\mathbb{R}^{3,1} \times T^2$ . In its DLCQ description, the  $(2,0)$  theory has an  $SU(N)$  gauge symmetry and generic states with momenta in the compact directions also carry the corresponding gauge charges. The gauge-invariant states of the  $\mathcal{N} = 4$  theory should correspond to the sector of the  $\sigma$ -model with  $Q_{eI} = Q_m^I = 0$ . We present two pieces of evidence in favour of this identification. First, we show that the supersymmetry of the  $\sigma$ -model is enlarged to an  $SU(1,1|4)$  superconformal invariance in this sector. This precisely matches the unbroken superconformal invariance of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory compactified on a light-like circle. In particular, the dilatation operator of the quantum mechanical model is related to the “light-cone” dilatation operator of the  $\mathcal{N} = 4$  theory. Second, in a weak-coupling limit where the curvature of the target space becomes small, the partons correspond to perturbative quanta of an  $SU(N)$  gauge theory. In the sector with  $Q_{eI} = Q_m^I = 0$ , the spectrum precisely coincides with that of the free  $\mathcal{N} = 4$  gauge theory on a light-like circle.

The results described above are provisional in character because the instanton moduli-space is singular. In addition to the familiar small instanton singularities, more severe singularities arise in the semi-flat limit relevant for describing the  $\mathcal{N} = 4$  theory. In order to obtain a useful description of  $\mathcal{N} = 4$  SUSY Yang-Mills we need a prescription for dealing with these singularities. To this end we consider a deformation of the  $\sigma$ -model corresponding to the introduction of a target space magnetic field. As we discuss in Section 5, the new model corresponds to a DLCQ description of the  $(2,0)$ -theory compactified on a certain  $T^2$  bundle over a four-dimensional pp-wave geometry. The deformation has no effect on states in the sector with zero parton momenta which should contain the states of the  $\mathcal{N} = 4$  theory. In particular it leaves the  $SU(1,1|4)$  superconformal symmetry of that sector unbroken. However, we will show that it has several interesting effects on generic states of the full model:

**1:** In the presence of the deformation, the full  $\sigma$ -model has a non-trivial superconformal invariance. In particular the new Hamiltonian is part of an  $\mathbf{osp}(1, 1|4)$  superconformal algebra. This is a subalgebra of the  $\mathbf{su}(1, 1|4)$  invariance of the sector of zero parton momenta.

**2:** The deformed theory is characterised by a new dimensionless parameter  $\rho$  measuring the strength of the applied magnetic field. In the sector with  $Q_{eI} = Q_m^I = 0$ , changing  $\rho$  simply corresponds to a change of basis of the conformal algebra which leaves the spectrum invariant. For  $Q_{eI}, Q_m^I \neq 0$ , the spectrum depends non-trivially on  $\rho$ .

**3:** For  $\rho \gg 1$ , wavefunctions of generic states are supported away from the singular points of  $\mathcal{M}$ . The resulting spectrum is discrete.

**4:** In the limit  $\rho \rightarrow \infty$ , for  $Q_{eI}, Q_m^I \neq 0$ , the model becomes (at least) semiclassically integrable. We determine the semiclassical spectrum of the model in this limit. At weak coupling, the spectrum can be related to that of an integrable spin chain.

In the limit  $\rho \rightarrow \infty$ , it seems possible that the model has full quantum integrability and that its spectrum can be determined exactly. The most interesting question, however, is what relation, if any, our results have to the integrability of the planar  $\mathcal{N} = 4$  theory. In the final section of the paper we make some preliminary remarks which should serve as a starting point for further investigation. In particular we note that, *after* taking  $\rho \rightarrow \infty$ , the resulting description in terms of an integrable system appears to extend smoothly to states with  $Q_{eI}, Q_m^I = 0$  suggesting that it may be possible to extract the operator dimensions of the  $\mathcal{N} = 4$  theory. This will be investigated elsewhere [7].

The paper is organised as follows. In Section 2, we review the properties of the relevant instanton moduli-space,  $\mathcal{M}$ . In Section 3 we construct the quantum mechanical  $\sigma$ -model with target space  $\mathcal{M}$  and its deformation by an external magnetic field. In Section 4, we describe the specific application to the DLCQ of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory. In Section 5 we propose a spacetime interpretation for the magnetic deformation. Finally in Section 6, we discuss prospects for extracting physical observables of the  $\mathcal{N} = 4$  theory from this approach.

## 2 The instanton moduli space

Following the proposal of [3, 4], we will consider supersymmetric quantum mechanics on the moduli space,  $\mathcal{M}$ , of  $K$  instantons of an  $SU(N)$  Yang-Mills theory living on  $\mathbb{R}^2 \times T^2$ . The torus has complex structure parameter  $\tau_{\text{cl}}$  and area  $\mathcal{A} = 4\pi^2 \text{Im}\tau_{\text{cl}} \ell^2$  where  $\ell$  is a parameter with the dimensions of length. We also introduce Wilson lines for the  $SU(N)$  gauge field on  $T^2$ . The Wilson line is specified by choosing  $N$  marked points on the dual torus  $\hat{T}^2$  (up to an overall translation). The instanton moduli space is a hyper-Kähler manifold and bosonic quantum mechanics on  $\mathcal{M}$  therefore admits a fermionic completion with  $\mathcal{N} = (4, 4)$  supersymmetry. In the remainder of this section we review several known results about the moduli space. The reader should consult the original references [4, 8, 9, 10, 11] for further details.

In fact the hyper-Kähler manifold  $\mathcal{M}$  arises in two distinct ways as the vacuum moduli space of an auxiliary supersymmetric gauge theory. First, the ADHM-Nahm construction of the moduli space can be interpreted as an infinite-dimensional hyper-Kähler quotient. The same quotient construction describes the Higgs branch of a  $U(K)$  gauge theory with eight supercharges on  $\hat{T}^2 \times \mathbb{R}^{2,1}$  with impurities localised at the  $N$  marked points on the torus. As the theory is three-dimensional in the IR, we can also apply the mirror symmetry of [12] to realise  $\mathcal{M}$  as the Coulomb branch of another theory with eight supercharges and three non-compact dimensions. We will call these two constructions the “Higgs branch” and “Coulomb branch” descriptions of  $\mathcal{M}$ . We will now briefly review both these descriptions.

### 2.1 The Coulomb branch description

The Coulomb branch description starts from an  $\mathcal{N} = 2$  quiver gauge theory in four dimensions. The corresponding quiver diagram coincides with the Dynkin diagram for the affine Lie algebra  $\hat{A}_{N-1}$ . The model coincides with the “elliptic quiver” introduced and solved in [13]. Thus we consider an  $\mathcal{N} = 2$  supersymmetric gauge theory with gauge group,

$$G = U(1)_D \times \prod_{j=1}^N SU(K)_j, \quad (2.1)$$

In addition to an  $\mathcal{N} = 2$  vector multiplet for each  $SU(K)$  factor in  $G$ , the theory contains hypermultiplets in the bifundamental representation of adjacent factors. Thus we have a hy-

permultiplet in the  $(\bar{\mathbf{K}}, \mathbf{K})$  of  $SU(K)_j \times SU(K)_{j+1}$  for  $j = 1, 2, \dots, N$  with the identification  $SU(K)_{N+1} \simeq SU(K)_1$ . All matter fields are neutral under a diagonal factor  $U(1)_D$  which decouples. The complexified gauge coupling of the diagonal  $U(K) = U(1)_D \times SU(K)_D / \mathbb{Z}_K$  coincides with the complex structure parameter  $\tau_{\text{cl}}$  of the torus in the ambient spacetime of the  $SU(N)$  instantons. The  $N - 1$  off-diagonal gauge couplings are encoded in the relative positions of  $N$  marked points on  $\hat{T}^2$ . The  $\beta$ -function of each gauge coupling vanishes and the four-dimensional theory has  $N$  exactly marginal couplings (including  $\tau_{\text{cl}}$ ).

The theory has a Coulomb branch where the complex scalars in the vector multiplet acquire non-zero vacuum expectation values and Higgs the gauge group down to its Cartan subgroup  $U(1)^r$  where  $r = KN - N + 1$ . The low-energy physics on the Coulomb branch is governed by a complex curve  $\Sigma$  of genus  $r$  [13]. To describe the curve we start with a canonical complex torus,

$$E(\tau) = \frac{\mathbb{C}}{2\omega_1\mathbb{Z} \oplus 2\omega_2\mathbb{Z}}$$

where  $\omega_1 = i\pi$  and  $\omega_2 = i\pi\tau_{\text{cl}}$ . We also specify  $N$  marked points  $z = Z_j$  for  $j = 1, 2, \dots, N$ . The gauge coupling of the factor  $SU(N)_j$  in  $G$  is identified as  $\tau_j = (Z_{j+1} - Z_j)/2\omega_1$  with the identification  $Z_{N+1} = Z_1$ .

The curve  $\Sigma$  is a  $K$ -fold branched cover of  $E(\tau)$  which is embedded in  $\mathbb{C} \times E(\tau)$  as follows. Choosing coordinates  $v \in \mathbb{C}$  and  $z \in E(\tau)$ ,  $\Sigma$  is specified by a polynomial equation of the form,

$$F(v, z) = v^K - f_1(z)v^{K-1} + f_2(z)v^{K-2} - \dots + (-1)^K f_K(z) = 0. \quad (2.2)$$

For each  $z \in E(\tau)$ , the equation for  $v$  has  $K$  roots corresponding to the  $K$  branches of the cover. The  $K$  coefficient functions are constrained as follows:  $f_1(z)$  is constant while  $f_a(z)$  for  $a > 1$  are meromorphic on  $E(\tau)$  with simple poles at the points  $z = Z_i$ , for  $i = 1, 2, \dots, N$  and no other singularities. We can solve these conditions in terms of elliptic functions<sup>5</sup> as follows,

$$\begin{aligned} f_1(z) &\equiv \mathcal{H}_1^{(0)} \\ f_a(z) &= \sum_{i=1}^N \mathcal{H}_a^{(i)} \xi(z - Z_i) + \mathcal{H}_a^{(0)} \quad \text{for } a = 2, 3, \dots, K. \end{aligned} \quad (2.3)$$

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<sup>5</sup>In the following  $\sigma(z)$ ,  $\xi(z) = \sigma'(z)/\sigma(z)$  and  $\mathcal{P}(z) = -\xi'(z)$  denote the standard Weierstrass functions

The  $(K-1)(N+1)+1$  complex coefficients  $\{\mathcal{H}_a^{(i)}, \mathcal{H}_a^{(0)}\}$  are subject to the constraints,

$$\sum_{i=1}^N \mathcal{H}_a^{(i)} = 0 \quad \text{for } a = 2, 3, \dots, K \quad (2.4)$$

and thus parametrise a moduli space  $\mathcal{N}(\Sigma)$  of complex dimension  $KN - N + 1$  which is identified with the Coulomb branch of the four-dimensional theory.

The low-energy physics on the Coulomb branch is determined by the periods of the differential  $\lambda = vdz$  which is meromorphic on  $\Sigma$ . We choose a canonical basis  $\{A^I, B_I\}$  of homology one-cycles on  $\Sigma$  obeying  $A^I \cap B_J = \delta_J^I$  for  $I, J = 1, 2, \dots, r$  and define corresponding periods,

$$a^I = \frac{1}{2\pi i} \oint_{A^I} \lambda \quad a_I^D = \frac{1}{2\pi i} \frac{\partial \mathcal{F}}{\partial a_I} = \frac{1}{2\pi i} \oint_{B_I} \lambda.$$

Here  $\mathcal{F}(a)$  is a holomorphic prepotential which completely determines the low-energy effective action. The bosonic part of the action takes the form

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \tau_{IJ} \partial_m a^I \partial^m \bar{a}^J + \frac{1}{8\pi} \text{Im} \tau_{IJ} v_{mn}^I v^{Jmn} + \frac{1}{8\pi} \text{Re} \tau_{IJ} v_{mn}^I \tilde{v}^{Jmn}, \quad (2.5)$$

where the matrix of low-energy  $U(1)^r$  gauge couplings is determined by the period matrix of  $\Sigma$ ,

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}. \quad (2.6)$$

The scalar part of the effective action is a nonlinear  $\sigma$ -model with target space  $\mathcal{N}(\Sigma)$  and the corresponding target space metric can be read from the scalar kinetic terms in (2.5);

$$ds^2 = \text{Im} \tau_{IJ} da^I d\bar{a}^J.$$

The existence of local coordinates  $\{a^I\}$  in which the metric is determined by a holomorphic prepotential in this way is the definition of special Kähler geometry. In particular, the moduli space  $\mathcal{N}(\Sigma)$  equipped with this metric is a special Kähler manifold. As the underlying quiver gauge theory is superconformal, the Coulomb branch metric is also scale-invariant. This implies that the prepotential satisfies a further condition,

$$a^I \frac{\partial}{\partial a^I} \mathcal{F} = 2\mathcal{F} \quad (2.7)$$

which will be important in the following.



The next step in the construction is to take the four-dimensional quiver gauge theory described above and compactify it down to three dimensions on a circle of radius  $R$ . The massless fields in the resulting three-dimensional effective theory include the massless scalars  $a^I$  of the four-dimensional theory. We also obtain  $r$  real scalars  $\theta_e^I$  from the Wilson lines of the massless  $U(1)^r$  gauge fields around the compact direction. A further  $r$  real scalars  $\theta_{m,I}$  arise after dualising the remaining components of the  $U(1)^r$  gauge field. The two sets of scalars are naturally periodic due to invariance under large gauge transformations and the quantisation of magnetic charge respectively. We normalise  $\theta_e^I$  and  $\theta_{m,I}$  to have period  $2\pi$  for all  $I$ .

The massless scalars  $\{a^I, \theta_e^I, \theta_{m,I}\}$  described above parametrise a Coulomb branch of real dimension  $4(KN - N + 1)$  which we identify with the instanton moduli space  $\mathcal{M}$ , where the compactification radius is identified as<sup>6</sup>  $R = 1/4\pi^2\ell^2$ . The low-energy effective action is a three-dimensional nonlinear  $\sigma$ -model which defines a metric on this space. As the theory has eight supercharges the resulting metric must be hyper-Kähler. It therefore has three linearly independent complex structures  $J^1, J^2$  and  $J^3$ . In fact there is a preferred complex structure [9],  $J = J^3$ , which is independent of the compactification radius  $R$ . In this complex structure, the holomorphic coordinates can be taken as  $\{a^I, z_I\}$  where  $a^I$  are the massless complex scalars of the four dimensional theory and,

$$z_I = \theta_{m,I} - \tau_{IJ}\theta_e^J$$

parametrise an  $r$ -dimensional complex torus which can be identified with the Jacobian,

$$\mathcal{J}(\Sigma) = \frac{\mathbb{C}^r}{\mathbb{Z}^r \oplus \tau\mathbb{Z}^r}.$$

In the preferred complex structure, the full Coulomb branch is therefore the total space of a fibre bundle<sup>7</sup>  $\mathcal{M} \rightarrow \mathcal{N}(\Sigma)$  over the special Kähler base  $\mathcal{N}(\Sigma)$  whose fibres are the Jacobian tori  $\mathcal{J}(\Sigma)$ . With respect to a given complex structure a hyper-Kähler manifold is a

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<sup>6</sup>Note the apparent mismatch in dimensions in this identification which can be explained as follows. In the context of a four-dimensional gauge theory compactified on  $S^1$ ,  $\mathcal{M}$  corresponds to a vacuum moduli space and the natural coordinates are scalar fields which have mass dimension one-half in three dimensions. Here we are working instead in coordinates with dimensions of length as appropriate for the application to instantons living on the spacetime  $\mathbb{R}^2 \times T^2$ .

<sup>7</sup>This statement is not quite precise due to an additional subtlety in the global definition of fibre coordinates known as the quadratic refinement [14]. However, this will not play a role in the following.

complex manifold with a holomorphic symplectic form. In the local holomorphic coordinates introduced above for  $J = J^3$ , the holomorphic symplectic form is given as,

$$\eta = g \cdot (J^1 + iJ^2) = da^I \wedge dz_I.$$

The hyper-Kähler metric on  $\mathcal{M}$  can be determined exactly using the methods of [14]. Although complicated in general, when  $R$  is much larger than any other scales in the problem the metric on the total space takes its *semi-flat* form:

$$G = R \operatorname{Im} \tau_{IJ} da^I d\bar{a}^J + \frac{1}{4\pi^2 R} (\operatorname{Im} \tau^{-1})^{IJ} \delta z_I \delta \bar{z}_J, \quad (2.8)$$

where the one-form,

$$\delta z_I = d\theta_{m,I} - \tau_{IJ} d\theta_e^J$$

is closed on the fibre  $\mathcal{J}(\Sigma)$  but not on the total space. The factor of  $1/R$  in front of the second term in the metric means that the volume of the fibre scales as  $1/R^r$ . In contrast, the factor  $R$  in front of the first term is conventional and can be removed by rescaling the coordinates  $a^I$ .

The semi-flat metric is valid up to instanton corrections of order  $\exp(-M_{\text{BPS}}R)$  where  $M_{\text{BPS}}$  corresponds to the mass of a BPS state in the four-dimensional quiver theory. To avoid later confusion we will call these 3d instantons. The four-dimensional BPS mass formula is  $M_{\text{BPS}} = |Z|$  where,

$$Z = n_I a^I + m^I a_I^D$$

and  $n_I$  and  $m^I$  are integer-valued electric and magnetic charges for  $I = 1, 2, \dots, r$ . Thus the semi-flat formula is valid in a limit where  $R \rightarrow \infty$  with  $a^I$  held fixed away from the special submanifolds in the moduli space where one or more of the cycles  $\{A^I, B_I\}$  degenerate. These are the famous Seiberg-Witten points of the four-dimensional theory where electric or magnetic BPS states become massless. The semi-flat metric itself has logarithmic singularities on these submanifolds which are smoothed out in the full metric by 3d instanton corrections<sup>8</sup>.

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<sup>8</sup>Note however that singularities of higher codimension remain where Higgs or mixed branches intersect the Coulomb branch.

Together with the diagonal coupling  $\tau_{cl}$ , the couplings  $\tau_j$ , for  $i = 1, 2, \dots, N$ , control the weak-coupling expansion of the 4d  $\mathcal{N} = 2$  theory underlying the Coulomb branch description of  $\mathcal{M}$ . Specifically the prepotential  $\mathcal{F}(a)$  can be expanded in the limit  $\tau_{cl}, \tau_j \rightarrow i\infty$  as,

$$\mathcal{F}(a) = \mathcal{F}_{cl} + \mathcal{F}_{1-loop} + \mathcal{F}_{inst}$$

where the three terms correspond to classical, one-loop and instanton contributions respectively in the 4d  $\mathcal{N} = 2$  theory. In the weak-coupling region, natural coordinates on the Coulomb branch are provided by the VEVs of the adjoint scalars  $\Phi_j$  in the vector multiplet of the  $SU(K)_j$  factor in the gauge group  $G$ :

$$\langle \Phi_j \rangle = \text{diag}(a_{j1}, a_{j2}, \dots, a_{jK}) \quad \text{for } j = 1, 2, \dots, N. \quad (2.9)$$

Here the coordinate index  $I$  has been traded for an  $SU(N)$  index,  $i, j = 1, 2, \dots, N$  and a  $U(K)$  index  $a, b = 1, 2, \dots, K$ . We can project out unwanted  $U(1)$  factors by imposing the constraint,

$$\sum_{b=1}^K a_{jb} = \bar{a} \quad \text{for } j = 1, 2, \dots, N \quad (2.10)$$

where  $\bar{a}$  is the scalar in the vector multiplet of the decoupled diagonal subgroup  $U(1)_D$ . In these coordinates we have,

$$\mathcal{F}_{cl} = \sum_{j=1}^N \sum_{b=1}^K \frac{1}{2} \tau_j a_{jb}^2 \quad (2.11)$$

$$\mathcal{F}_{1-loop} = -\frac{1}{2\pi i} \sum_{j=1}^N \sum_{a>b}^K f(a_{ja} - a_{jb}) + \frac{1}{4\pi i} \sum_{j=1}^N \sum_{a,b=1}^K f(a_{ja} - a_{j+1b}) \quad (2.12)$$

where  $f(x) = x^2(2\log x - 1)$ . In the 4d theory, the first term in (2.12) is the one-loop contribution from integrating out the massive states of the vector multiplets while the second comes from the bifundamental hypermultiplets. Terms in the instanton contribution  $\mathcal{F}_{inst}$  are suppressed by powers of  $\exp(2\pi i \tau_{cl})$  and/or  $\exp(2\pi i \tau_j)$ . We will call these effects, which do contribute to the semi-flat metric, 4d instantons to distinguish them from the 3d instantons discussed above which do not.

In particular, for  $\text{Im}\tau_{cl} \gg 1$  and  $\text{Im}\tau_j \gg 1$ , we can approximate the prepotential  $\mathcal{F}(a)$  by  $\mathcal{F}_{cl}(a)$ . The resulting moduli space metric is flat. Taking account of the action of the

Weyl group in each  $SU(K)$  factor and of the constraint (2.15) the instanton moduli space reduces to,

$$\mathcal{M}_{\text{cl}} = \mathbb{R}^2 \times T^2[\tau_{\text{cl}}] \times \prod_{j=1}^N \frac{\text{Sym}^K(\mathbb{R}^2 \times T^2[\tau_j])}{\mathbb{R}^2 \times T^2[\tau_j]}$$

with a flat metric. Here  $T^2[\tau]$  is a two-dimensional torus of complex structure  $\tau$  and  $\text{Sym}^K$  denotes a  $K$ -fold symmetric product. The area of each  $T^2$  factor is proportional to  $1/R \sim \ell^2$ . This form reflects a feature which is familiar from related studies of instantons on tori (see especially [15]);  $K$  instantons of gauge group  $SU(N)$  fractionate into  $KN$  “partons” each of which carries a fraction of the total instanton charge. The resulting configuration consists of  $K$  identical partons of  $N$  distinct species. A less familiar feature is that the  $N - 1$  relative positions of the centre of mass on  $\mathbb{R}^2 \times T^2$  of each of the  $N$  distinct species of parton are frozen.

The one-loop correction coming from  $\mathcal{F}_{1\text{-loop}}$  introduces logarithmic singularities in the semi-flat metric. In particular singularities occur where electrically charged particles from the vector and hyper-multiplets become light. The inclusion of instantons corrects the picture further, resulting in loci where magnetically charged states become light.

## 2.2 The Higgs branch description

The Higgs branch description of the hyper-Kähler manifold  $\mathcal{M}$  starts from the ADHM-Nahm transform for  $K$   $SU(N)$  instantons on  $\mathbb{R}^2 \times T^2$  with Wilson lines [4]. This results in a  $U(K)$  gauge theory on the dual torus  $\hat{T}^2$  which, up to a rescaling of the area, we identify with the canonical torus  $E(\tau)$ . The resulting model has localised impurities at the  $N$  points  $z = Z_i$  specified by the Wilson lines. The theory contains a  $U(K)$  gauge field with components  $A_z(z, \bar{z})$  and  $A_{\bar{z}}(z, \bar{z})$  as well as a complex scalar field  $\phi(z, \bar{z})$  in the adjoint of the gauge group. It also contains localised impurities  $\{Q_i, \tilde{Q}_i\}$  in the  $\mathbf{K} \oplus \bar{\mathbf{K}}$  of  $U(K)$  defined at the marked points  $z = Z_i$  on  $\hat{T}^2$  for  $i = 1, 2, \dots, N$ . In general, we construct the Higgs branch by imposing the F- and D-term equations of the theory and modding out by the  $U(K)$  gauge symmetry. In order to describe  $\mathcal{M}$  as a complex symplectic manifold (in the preferred complex structure  $J$ ) it suffices instead to impose only the F-term equation which

reads,

$$\partial_{\bar{z}}\phi + [A_{\bar{z}}, \phi] = 2\pi i \sum_{i=1}^N Q_i \tilde{Q}_i \delta^{(2)}(z - Z_i) \quad (2.13)$$

and divide out by the complexified gauge symmetry [16],

$$\begin{aligned} A_z &\rightarrow \Omega A_z \Omega^{-1} - \Omega \partial_z \Omega^{-1} \\ \phi &\rightarrow \Omega \phi \Omega^{-1} \quad \Omega(z, \bar{z}) \in GL(K, \mathbb{C}). \end{aligned} \quad (2.14)$$

Following the analysis of [8, 11], we first use the off-diagonal part of the complex gauge symmetry to bring  $A_{\bar{z}}$  to the diagonal form,

$$A_{\bar{z}} = \frac{\pi i}{2(\bar{\omega}_2 \omega_1 - \bar{\omega}_1 \omega_2)} \text{diag}(X_1, X_2, \dots, X_K).$$

The remaining degrees of freedom  $X_a$  have a natural periodic identification,

$$X_a = X_a + 2n\omega_1 + 2m\omega_2 \quad m, n \in \mathbb{Z}$$

corresponding to large gauge transformations. In this gauge we may solve (2.13) explicitly for  $\phi(z, \bar{z})$ . For the diagonal elements we find,

$$\phi_{aa}(z) = P_a + \sum_{i=1}^N Q^{ia} \tilde{Q}^{ia} \xi(z - Z_i)$$

where  $P_a$ , for  $a = 1, 2, \dots, K$ , are undetermined complex constants. The off-diagonal elements are given as,

$$\phi_{ab}(z) = \sum_{i=1}^N Q^{ia} \tilde{Q}^{ib} \frac{\sigma(X_{ab} + z - Z_i)}{\sigma(X_{ab}) \sigma(z - Z_i)}$$

for  $a, b = 1, 2, \dots, K$  with  $a \neq b$  where  $X_{ab} = X_a - X_b$  and  $\sigma(z)$  is the Weierstrass  $\sigma$ -function.

The resulting description of the complex manifold  $\mathcal{M}$  is given in holomorphic coordinates provided by the  $2KN + 2K$  complex parameters  $\{Q^{ia}, \tilde{Q}^{ia}, X^a, P^a\}$  subject to the  $K + N - 1$  constraints<sup>9</sup>,

$$\sum_{a=1}^K Q^{ia} \tilde{Q}^{ia} = 0 \quad \sum_{i=1}^N Q^{ia} \tilde{Q}^{ia} = 0 \quad (2.15)$$

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<sup>9</sup>Here the first set of  $N - 1$  independent constraints in (2.15) and their accompanying gauge transformations in (2.16) correspond to the restriction from  $U(K)$  to  $SU(K)$  in each factor of the Coulomb branch gauge group  $G$ . The second set of  $K$  constraints are implied by the diagonal components of equation (2.13), while the accompanying gauge transformations correspond to the diagonal generators of  $GL(K, \mathbb{C})$ .

and the residual gauge symmetry,

$$\begin{aligned} Q_{ia} &\rightarrow \xi_a Q_{ia} \xi_i \\ \tilde{Q}_{ia} &\rightarrow \xi_a^{-1} \tilde{Q}_{ia} \xi_i^{-1} \quad \xi_a, \xi_i \in \mathbb{C} \end{aligned} \quad (2.16)$$

of dimension  $K + N - 1$  thus giving a complex manifold of the expected complex dimension  $2(KN - N + 1)$ .

The above construction is an infinite-dimensional version of the standard hyper-Kähler quotient. In the preferred complex structure it provides a quotient construction of the complex symplectic manifold  $(\mathcal{M}, J, \eta)$ . In particular, one starts from the infinite-dimensional space  $\mathcal{M}_\infty$ , spanned by the fields of the 2d theory  $\{\phi(z, \bar{z}), A_z(z, \bar{z}), Q^i, \tilde{Q}^i\}$  endowed with the holomorphic symplectic form,

$$\eta_\infty = \int d^2z \sum_{a,b=1}^K d\phi_{ab}(z, \bar{z}) \wedge dA_{ba}^z(z, \bar{z}) + \sum_{i=1}^N \sum_{a=1}^K dQ^{ai} \wedge d\tilde{Q}^{ai}.$$

After imposing the F-term (2.13) equation, the symplectic form descends to the quotient space where it takes the form,

$$\eta = \sum_{a=1}^K dX^a \wedge dP^a + \sum_{i=1}^N \sum_{a=1}^K dQ^{ia} \wedge d\tilde{Q}^{ia}.$$

Finally the constraints (2.15) can be imposed and the diagonal complex gauge transformations (2.16) divided out in a further finite-dimensional symplectic quotient.

The symplectic quotient construction also provides a tower of Poisson-commuting Hamiltonians which promote  $(\mathcal{M}, J, \eta)$  to an algebraic integrable system. Here we define the Poisson bracket of any two holomorphic functions  $f$  and  $g$  on  $\mathcal{M}$  as,

$$\{f, g\} = (\eta^{-1})^{uv} \frac{\partial f}{\partial w^u} \frac{\partial g}{\partial w^v}$$

where  $w^u$ ,  $u = 1, 2, \dots, 2r$  are holomorphic coordinates on  $\mathcal{M}$ . The quotient construction guarantees that all quantities which are invariant under complex gauge transformation (2.14) and which Poisson-commute on the big phase space  $\mathcal{M}_\infty$  will descend to Poisson-commuting quantities on the quotient space. In particular we have,

$$\{\mathrm{Tr}_K \phi^l(z), \mathrm{Tr}_K \phi^m(z')\} = 0 \quad \forall l, m \in \mathbb{Z} \quad \forall z, z' \in E(\tau).$$

The tower of conserved charges is encoded in the *spectral curve* of the integrable system,

$$F(v, z) = \det(v\mathbb{I}_K - \phi(z)) = 0. \quad (2.17)$$

Examination of the singularities of  $F(v, z)$  reveals that it obeys the same constraints as the function of the same name appearing in (2.2). Thus the spectral curve is precisely the same as the complex curve  $\Sigma$  which appears in the Coulomb branch description of the moduli space. Hence using the parametrisation (2.3) we find that the quantities  $\{\mathcal{H}_a^{(i)}, \mathcal{H}_a^{(0)}\}$  subject to the constraint (2.4) provide a tower of  $KN - N + 1$  commuting Hamiltonians on the complex phase space  $(\mathcal{M}, J, \eta)$ .

When expressed in terms of the Higgs branch coordinates,  $\{Q^{ia}, \tilde{Q}^{ia}, X^a, P^a\}$ , the integrable system takes the form of a holomorphic many-body problem [8]. In particular, the symplectic form yields Poisson brackets,

$$\{X_a, P_b\} = \delta_{ab} \quad \{S_{ij}^a, S_{kl}^b\} = \delta_{ab} (\delta_{jk} S_{il}^a - \delta_{il} S_{kj}^a) \quad (2.18)$$

where we have defined spin variables,

$$S_{ij}^a = Q_i^a \tilde{Q}_j^a \quad (2.19)$$

for  $a = 1, 2, \dots, K$  which obey the  $SL(N, \mathbb{C})$  Poisson algebra as given in (2.18). Explicit expressions for the Hamiltonians can be found in [11]. In the simplest case where all  $K$  punctures coincide,  $Z_i = Z_j$  for all  $i, j = 1, 2, \dots, N$ , one particular quadratic combination of the Hamiltonians takes the form,

$$H_2 = \sum_{a=1}^K P_a^2 + \sum_{a \neq b} \sum_{i,j=1}^N S_{ij}^a S_{ji}^b \mathcal{P}(X_a - X_b) \quad (2.20)$$

where  $\mathcal{P}(z)$  is the Weierstrass elliptic function. This is the Hamiltonian for a generalisation of the elliptic Calogero-Moser model in which each of the  $K$  particles carries an  $SL(N, \mathbb{C})$  spin. The integrability of this model was used to solve it exactly in [17].

Another interesting case is the limit  $\tau \rightarrow i\infty$ , with  $\tau_j$  held fixed, where the elliptic quiver degenerates to a linear quiver based on the Dynkin diagram of  $A_{N-1}$ . In the simplest case  $N = 2$ , the resulting theory is  $\mathcal{N} = 2$  SUSY QCD with gauge group  $U(K)$  and  $2K$  massless hypermultiplets in the fundamental representation. In this limit, the variables  $P_a$  describing

the momenta of the Calogero particles are frozen and the conjugate variables  $X_a$  are gauged away. The resulting Hamiltonians for the spins  $S_{ij}^a$  describe a classical Heisenberg spin chain with  $SL(N, \mathbb{C})$  spins at  $K$  sites [18, 19]. The classical Casimirs of the  $SL(N, \mathbb{C})$  Poisson algebra (2.18) vanish at each site and the off-diagonal couplings  $\tau_j$  correspond to twisted boundary conditions for the chain.

The integrable system provides a nice way of understanding the equivalence between the Higgs and Coulomb branch descriptions of  $(\mathcal{M}, J, \eta)$ . As we have already noted above, the spectral curve of the integrable system coincides with the Seiberg-Witten curve  $\Sigma$  of the Coulomb branch description. The commuting Hamiltonians of the integrable system coincide with the moduli of the curve which parametrise the base of the Jacobian fibration in the Coulomb branch picture. We then recognise the coordinates  $\{a^I, z_I\}$  as the action-angle variables of the system. Indeed, the solution to the model given in [17] precisely takes the form of a linear flow on the Jacobian  $\mathcal{J}(\Sigma)$ .

### 3 The $\sigma$ -model

We now wish to consider a quantum mechanical  $\sigma$ -model whose target space is the instanton moduli space  $\mathcal{M}$  described in the previous section with the semi-flat metric (2.8). We work in local holomorphic coordinates  $\{a^I, z_I\}$ , with  $I = 1, 2, \dots, r$  where, as above,

$$z_I = \theta_{mI} - \tau_{IJ} \theta_e^J.$$

As the target space is hyper-Kähler, the bosonic  $\sigma$ -model action has a completion with  $\mathcal{N} = (4, 4)$  supersymmetry [20] and a  $USp(4) \simeq SO(5)$  R-symmetry [21, 22].

The resulting supersymmetric  $\sigma$ -model contains fermions<sup>10</sup>  $\psi^{IA}$ ,  $\bar{\psi}^{\bar{I}\bar{A}}$  in the  $\mathbf{4}$  and  $\bar{\mathbf{4}}$  of  $USp(4)$  respectively in addition to the bosonic coordinates  $a^I$ ,  $z_I$ . The couplings in the  $\sigma$ -model Lagrangian are completely determined by the holomorphic prepotential  $\mathcal{F}(a)$  introduced above. In the following,  $\mathcal{F}_{IJK}^{(3)}$  and  $\mathcal{F}_{IJKL}^{(4)}$  denote the third and fourth mixed partial derivatives of the prepotential with respect to the base coordinates  $a^I$ . The Lagrangian reads

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<sup>10</sup>In this section we will adopt the standard convention that all anti-holomorphic coordinate indices are barred.



[6],

$$\begin{aligned}\mathcal{L}_\sigma &= g_{I\bar{J}} \dot{a}^I \dot{\bar{a}}^{\bar{J}} + g^{I\bar{J}} \frac{\delta z_I}{\delta t} \frac{\delta \bar{z}_{\bar{J}}}{\delta t} \\ &\quad + ig_{I\bar{J}} \bar{\psi}_A^{\bar{J}} D_t \psi^{IA} - \frac{1}{12} \text{Re} [\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}] \\ &\quad - \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}_A^{\bar{J}} \psi^{KB} \bar{\psi}_B^{\bar{L}}.\end{aligned}\tag{3.1}$$

where,

$$\begin{aligned}D_t \psi^{IA} &= \dot{\psi}^{IA} - \frac{i}{2} (\text{Im } \tau^{-1})^{IL} \mathcal{F}_{LJK}^{(3)} \dot{a}^J \psi^{KA} \\ \frac{\delta z_I}{\delta t} &= \dot{z}_I - \mathcal{F}_{IJK}^{(3)} (\text{Im } \tau^{-1})^{KL} \text{Im } z_L \dot{a}^J - \frac{1}{4} \mathcal{F}_{IJK}^{(3)} \Omega_{AB} \psi^{JA} \psi^{KB}\end{aligned}\tag{3.2}$$

with  $\Omega_{AB}$  being the invariant antisymmetric tensor of  $USp(4)$ , and,

$$R_{I\bar{J}K\bar{L}} = -\frac{1}{4} (\text{Im } \tau^{-1})^{M\bar{N}} \mathcal{F}_{IKM}^{(3)} \bar{\mathcal{F}}_{\bar{J}\bar{L}\bar{N}}^{(3)}\tag{3.3}$$

denotes the base space Riemann tensor. We also define the totally symmetric base space tensor,

$$G_{IJKL} = -\frac{i}{2} \mathcal{F}_{IJKL}^{(4)} + \frac{1}{4} (\text{Im } \tau^{-1})^{MN} \left( \mathcal{F}_{ILM}^{(3)} \mathcal{F}_{JKN}^{(3)} + \mathcal{F}_{JLM}^{(3)} \mathcal{F}_{IKN}^{(3)} + \mathcal{F}_{KLM}^{(3)} \mathcal{F}_{IJN}^{(3)} \right).\tag{3.4}$$

The Hamiltonian formulation of the  $\sigma$ -model was worked out in detail in [6]. The dynamical variables are the base coordinates  $a^I$  and their ‘‘covariant’’ conjugate momenta  $\Pi_I = \text{Im } \tau_{I\bar{J}} \dot{\bar{a}}^{\bar{J}}$ , the fibre coordinates  $z_I$  and their canonical conjugate momenta  $P^I = \delta \mathcal{L}_\sigma / \delta \dot{z}_I$  as well as the fermions  $\psi^{IA}$  and  $\bar{\psi}^{\bar{I}\bar{A}}$ . The relevant non-zero (anti-)commutation relations are,

$$\begin{aligned}[a^I, \Pi_J] &= i\delta_J^I \\ [z_I, \Pi_J] &= i \text{Im } z_K \mathcal{F}_{IJL}^{(3)} (\text{Im } \tau^{-1})^{KL} & [z_I, P^J] &= i\delta_I^J \\ [\Pi_I, P^J] &= \frac{1}{2} \mathcal{F}_{IKL}^{(3)} (\text{Im } \tau^{-1})^{JL} P^K & [\Pi_I, \bar{P}^{\bar{J}}] &= -\frac{1}{2} \mathcal{F}_{IKL}^{(3)} (\text{Im } \tau^{-1})^{\bar{J}L} P^K\end{aligned}$$

and

$$\begin{aligned}[\Pi_I, \psi^{JA}] &= i\Gamma_{IK}^J \psi^{KA}, & [\Pi_I, \bar{\psi}^{\bar{J}\bar{A}}] &= 0 \\ \{\psi^{IA}, \bar{\psi}^{\bar{J}\bar{B}}\} &= \delta^{A\bar{B}} (\text{Im } \tau^{-1})^{I\bar{J}}.\end{aligned}$$

In terms of these variables, the  $\sigma$ -model Hamiltonian reads,

$$\begin{aligned}
H = & (\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}} + \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}^{\bar{J}} \psi^{KB} \bar{\psi}^{\bar{L}} \\
& + \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) \\
& + \text{Im } \tau_{I\bar{J}} P^I \bar{P}^{\bar{J}} + \frac{1}{2} \text{Re} \left( \mathcal{F}_{IJK}^{(3)} \Omega_{AB} \psi^{JA} \psi^{KB} P^I \right)
\end{aligned}$$

and the supercharges,

$$\begin{aligned}
Q^A = & \psi^{IA} \Pi_I + \frac{1}{12} \epsilon^A_{\bar{B}\bar{C}\bar{D}} \bar{\mathcal{F}}_{\bar{I}\bar{J}\bar{K}}^{(3)} \bar{\psi}^{\bar{I}\bar{B}} \bar{\psi}^{\bar{J}\bar{C}} \bar{\psi}^{\bar{K}\bar{D}} \\
& + \text{Im } \tau_{I\bar{J}} P^I \Omega^A_{\bar{B}} \bar{\psi}^{\bar{J}\bar{B}}
\end{aligned}$$

and  $\bar{Q}^{\bar{A}} = (Q^A)^\dagger$  transform in the  $\mathbf{4} \oplus \bar{\mathbf{4}}$  of  $USp(4)$ , commute with  $H$ , and obey the supersymmetry algebra,

$$\begin{aligned}
\{Q^A, Q^{\bar{B}}\} &= \delta^{A\bar{B}} H \\
\{Q^A, Q^B\} &= \{Q^{\bar{A}}, Q^{\bar{B}}\} = 0.
\end{aligned}$$

A striking feature of the semi-flat  $\sigma$ -model Lagrangian (3.1) is the presence of  $2(KN - N + 1)$   $U(1)$  global symmetries corresponding to constant shifts of the real fibre coordinates  $\theta_e, \theta_m$ . In the Hamiltonian formalism these lead to the Noether charges,

$$Q_{eI} = \tau_{IJ} P^J + \bar{\tau}_{IJ} \bar{P}^{\bar{J}} \quad Q_m^I = P^I + \bar{P}^{\bar{I}}. \quad (3.6)$$

One may readily check that these charges are conserved and also commute with each other. Although the charges are related to the conserved quantities of the integrable system described in the previous section it is important to note that the full  $\sigma$ -model is certainly not integrable. The number of real conserved charges is  $2r = 2(KN - N + 1)$  which is half the dimension of the target space and therefore half the number needed for integrability. However it will be useful to label states in the  $\sigma$ -model by the corresponding eigenvalues of these conserved quantities.

In particular, we will start by focusing on states carrying zero momentum in the fibre directions,  $Q_e = Q_m = 0$ , or equivalently  $P = 0$ . This sector of the theory has an enhanced conformal symmetry in which the Hamiltonian  $H$  is joined by a dilatation operator and

special conformal generator,

$$D = a^I \Pi_I + \bar{a}^{\bar{I}} \bar{\Pi}_{\bar{I}}$$

$$K = \text{Im} \left( \frac{\partial \mathcal{F}}{\partial a^I} \bar{a}^I \right)$$

which form an  $SL(2, \mathbb{R}) \simeq SO(2, 1)$  conformal algebra,

$$[H, K] = -iD \quad [D, K] = -2iK \quad [D, H] = 2iH.$$

Further the  $SO(5)$  R-symmetry of the full  $\sigma$ -model is enhanced to  $U(4) \simeq U(1) \times SU(4)$  with generators,

$$\mathcal{R} = i \left( a^I \Pi_I - \bar{a}^{\bar{I}} \bar{\Pi}_{\bar{I}} \right) + \frac{1}{2} \text{Im} \tau_{I\bar{J}} \psi^{IA} \bar{\psi}_A^{\bar{J}}$$

$$R^{A\bar{B}} = i \text{Im} \tau_{I\bar{J}} \left( \psi^{IA} \bar{\psi}^{\bar{J}\bar{B}} - \frac{1}{4} \delta^{A\bar{B}} \psi^{IC} \bar{\psi}_C^{\bar{J}} \right). \quad (3.8)$$

The supercharges  $Q^A$  and  $\bar{Q}^{\bar{A}}$  which now transform in the  $\mathbf{4} \oplus \bar{\mathbf{4}}$  of  $U(4)$  are supplemented by the superconformal generators,

$$S^A = \text{Im} \tau_{I\bar{J}} \bar{a}^{\bar{J}} \psi^{IA} \quad \bar{S}^{\bar{A}} = (S^A)^\dagger$$

also in the  $\mathbf{4} \oplus \bar{\mathbf{4}}$  of  $U(4)$ . The generators listed above close onto an  $\mathbf{su}(1, 1|4)$  superalgebra of conformal transformations [6]. This is precisely the superconformal symmetry algebra of  $\mathcal{N} = 4$  supersymmetric Yang-Mills compactified on a null circle.

The spectrum of states with zero momentum in the fibre directions can be decomposed in irreducible representations of the light-cone superconformal symmetry  $SU(1, 1|4)$ . In the context of the application to DLCQ, an obvious goal for our analysis is to try to determine this spectrum or equivalently determine the spectrum of the dilatation operator  $D$ . For many purposes it is convenient to perform a similarity transformation to a different basis of generators for the superconformal group [23, 24]. Let us consider the following family of generators labelled by a parameter  $\mu$  with the dimensions of mass:

$$\mathcal{L}_0[\mu] = \frac{1}{\mu} H + \mu K$$

$$\mathcal{L}_\pm[\mu] = \frac{1}{2} \left( \frac{1}{\mu} H - \mu K \pm iD \right)$$

which obey the  $\mathfrak{sl}(2, \mathbb{R})$  commutation rules in the standard form,

$$[\mathcal{L}_0, \mathcal{L}_\pm] = \pm 2\mathcal{L}_\pm \quad [\mathcal{L}_+, \mathcal{L}_-] = -2\mathcal{L}_0.$$

Under the similarity transformation, the problem of diagonalising  $D$  can be mapped onto the equivalent problem of diagonalising  $\mathcal{L}_0$ ;

$$\text{Spec}(D) = \text{Spec}(\mathcal{L}_0[\mu]) \quad \forall \mu \in \mathbb{R}/\{0\}.$$

More explicitly we have,

$$\mathcal{L}_0[\mu] = \frac{1}{\mu} (\text{Im } \tau^{-1})^{IJ} \Pi_I \bar{\Pi}_J + \mu \text{Im } \tau_{IJ} a^I \bar{a}^J + \text{fermions}.$$

Thus we see that the special conformal generator  $K$  contributes a harmonic potential centered at the origin  $a^I = 0$ . It follows that eigenstates of  $\mathcal{L}_0$  are supported near this point which, unfortunately, is highly singular. Indeed all the singular submanifolds where one or more cycles of  $\Sigma$  degenerate intersect at the origin. To make further progress we either need to find a way of resolving the singularity or restrict our attention to states whose wavefunctions are supported away from the singularities. In fact we will focus on the latter option.

We now consider generic states with non-zero momenta along the fibre directions. Naïvely the scale set by the volume of the fibre breaks superconformal invariance. Nevertheless we find that the operator  $\mathcal{L}_0$  has a non-trivial “lift” to the full theory. We define the operator,

$$\mathbb{H}[\mu] = \frac{1}{\mu} H + \mu K - \text{Im } \tau_{IJ} P^I \bar{a}^J - \text{Im } \tau_{IJ} \bar{P}^I a^J$$

which reduces to the  $SL(2, \mathbb{R})$  conformal generator  $\mathcal{L}_0[\mu]$  when the fibre momentum vanishes. In fact, for each value of  $\mu$  we find that  $\mathbb{H}[\mu]$  is part of a non-trivial  $OSp(1, 1|4)$  subgroup of the  $SU(1, 1|4)$  superconformal invariance which survives in the full theory. To exhibit this, we define supercharges,

$$\mathbb{Q}_+^A = q_+^A + i\Omega_{\bar{B}}^A \bar{q}_+^{\bar{B}} \quad \mathbb{Q}_-^{\bar{A}} = \bar{q}_-^{\bar{A}} - i\Omega_{\bar{B}}^{\bar{A}} q_-^B$$

where,

$$q_\pm^A = \frac{1}{\sqrt{\mu}} Q^A \pm i\sqrt{\mu} S^A \quad \bar{q}_\pm^{\bar{A}} = \frac{1}{\sqrt{\mu}} \bar{Q}^{\bar{A}} \pm i\sqrt{\mu} \bar{S}^{\bar{A}}$$

and  $USp(4)$  R-symmetry generators,

$$\mathbb{R}^{A\bar{A}} = R^{A\bar{A}} - \Omega_{\bar{B}}^A \Omega_{\bar{B}}^{\bar{A}} R^{B\bar{B}}.$$

These generators form an  $OSp(1, 1|4)$  super-algebra with non-vanishing (anti-)commutators,

$$[\mathbb{H}, \mathbb{Q}_+^A] = \mathbb{Q}_+^A \quad [\mathbb{H}, \mathbb{Q}_-^{\bar{A}}] = -\mathbb{Q}_-^{\bar{A}}$$

and

$$\{\mathbb{Q}_+^A, \mathbb{Q}_-^{\bar{A}}\} = 2\delta^{A\bar{A}}\mathbb{H} + 2i\mathbb{R}^{A\bar{A}}.$$

We omit the commutators for  $\mathbb{R}^{A\bar{A}}$  with the other generators as these are dictated by assignment of  $\mathbb{Q}_+$  and  $\mathbb{Q}_-$  to  $USp(4)$  representations  $\mathbf{4}$  and  $\bar{\mathbf{4}}$  respectively. Despite the fact that the  $SL(2, \mathbb{R})$  invariance of the zero momentum sector is broken, the surviving subgroup preserves some of the important features of superconformal invariance. In particular, the supercharges do not commute with  $\mathbb{H}$ , instead they have eigenvalues  $\pm 1$  under commutation with this operator. In addition, the  $USp(4)$  R-symmetry arises in the commutators of the supercharges and is therefore part of the algebra rather than an outer automorphism.

To further motivate the definition of the new operator we consider its form in more detail,

$$\mathbb{H}[\mu] = \frac{1}{\mu} (\text{Im } \tau^{-1})^{IJ} \Pi_I \bar{\Pi}_J + \frac{1}{\mu} \text{Im } \tau_{IJ} (P^I - \mu a^I) (\bar{P}^J - \mu \bar{a}^J) + \text{fermions}. \quad (3.9)$$

Hence we see that the harmonic potential contributing to  $\mathcal{L}_0$  has been translated and now has its centre at,

$$a^I = \frac{1}{\mu} P^I$$

for  $I = 1, 2, \dots, r$ , and it follows that eigenfunctions of  $\mathbb{H}$  are exponentially localised near this point. As the  $P^I$  (or more precisely the real quantities  $Q_e$  and  $Q_m$  defined above) are conserved quantities, we can set them to arbitrary fixed values (at least classically) and the condition picks out a single point on the base. For such generic values, the metric on  $\mathcal{M}$  is non-singular. Hence we expect that the operator  $\mathbb{H}$  has a discrete spectrum with non-singular eigenfunctions.

What is the physical interpretation of  $\mathbb{H}$ ? First, note that, up to a rescaling,  $\mathbb{H}$  differs from the  $\sigma$ -model Hamiltonian  $H$  only by a shift of the canonical fibre momentum<sup>11</sup>,  $P^I \rightarrow$

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<sup>11</sup>To check this statement for the fermionic sector of the model one needs to use the scale invariance condition (2.7) for the prepotential.

$$P^I - \mu a^I;$$

$$\mathbb{H} = \frac{1}{\mu} H \Big|_{P^I \rightarrow P^I - \mu a^I}.$$

As the original model describes a free particle moving on the instanton moduli space  $\mathcal{M}$ , the modification has a natural interpretation as coupling to a background vector potential specified by the one-form,

$$\mathcal{A} = \mu a^I dz_I + \mu \bar{a}^I d\bar{z}_I$$

which corresponds to a target space magnetic field,

$$\mathcal{F} = \mu da^I \wedge dz_I + \mu d\bar{a}^I \wedge d\bar{z}_I = \mu (\eta + \bar{\eta})$$

proportional to (the real part of) the globally-defined holomorphic symplectic form  $\eta$ . Equivalently we can pass back to the Lagrangian formulation and work with a new Lagrangian,

$$\begin{aligned} \tilde{\mathcal{L}}_\sigma &= \mathcal{L}_\sigma + \delta\mathcal{L} \\ &= \mathcal{L}_\sigma - \mu a^I \dot{z}_I - \mu \bar{a}^I \dot{\bar{z}}_I \end{aligned}$$

## 4 Discrete light-cone quantisation

Finally we are ready to discuss the main physical application for the quantum mechanical  $\sigma$ -model developed above: the description of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory in discrete light-cone quantisation (DLCQ). More precisely we will consider the six-dimensional  $(2, 0)$  superconformal field theory of Type  $A_{N-1}$  compactified to four dimensions on a torus  $T^2$  of complex structure  $\tau_{\text{cl}}$  and area  $\mathcal{A} = 4\pi^2 \text{Im}\tau_{\text{cl}} \ell^2$ . The compactified theory flows in the IR to  $\mathcal{N} = 4$  SUSY Yang-Mills with gauge group  $SU(N)$  and complexified coupling,

$$\tau_{\text{cl}} = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}.$$

The  $\mathcal{N} = 4$  theory is defined on four-dimensional Minkowski space  $\mathbb{R}^{1,3}$  with coordinates  $\{x_0, x_1, x_2, x_3\}$ . In addition to the degrees of freedom of  $\mathcal{N} = 4$  SUSY Yang-Mills, the full theory contains two towers of Kaluza-Klein states, with masses  $\sim 1/\ell$ , carrying non-zero momentum on  $T^2$ .

To study the above theory in DLCQ, we will compactify the null direction  $x_- = x_0 - x_1$  on a circle of radius  $R_-$ . The conjugate momentum is therefore quantised:  $p_+ = K/R_-$  with  $K \in \mathbb{N}$ . According to the proposal of [3, 4], the sector of the theory with  $K$  units of null momentum is described by maximally supersymmetric quantum mechanics on the moduli space of  $K$  Yang-Mills instantons of an auxiliary  $SU(N)$  gauge theory on  $\mathbb{R}^2 \times T^2$ . This coincides with the quantum mechanical  $\sigma$ -model with target space  $\mathcal{M}$  described in the previous sections. The remaining non-compact light-cone coordinate  $x_+ = x_0 + x_1$  plays the role of time and the conjugate momentum is identified with the  $\sigma$ -model Hamiltonian. Restoring appropriate dimensionful factors we have,

$$\begin{aligned} p_- &= H \\ &= R_- \text{Im} \tau_{\text{cl}} \left[ (\text{Im} \tau^{-1})^{IJ} \Pi_I \bar{\Pi}_J + \frac{4}{\ell^2} \text{Im} \tau_{IJ} P^I \bar{P}^J + \text{fermions} \right]. \end{aligned} \quad (4.1)$$

Here we have rescaled the base coordinates  $a^I$  so that they have the dimensions of length as appropriate for the DLCQ interpretation. More precisely, this  $\sigma$ -model Hamiltonian (4.1) describes physics in the region of the moduli space where the metric can be replaced by its semi-flat form (2.8). Roughly speaking, this corresponds to studying the theory at length-scales much larger than the compactification scale  $\ell$ . This region should be the relevant one for understanding the DLCQ description of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory.

In addition to  $\tau_{\text{cl}}$ ,  $R_-$  and  $\ell$ , the  $\sigma$ -model also depends on  $N - 1$  complex parameters corresponding to the relative positions of  $N$  marked points  $z = Z_j$  on a torus of complex structure  $\tau_{\text{cl}}$ . In DLCQ, these parameters correspond to a complex combination of light-like electric and magnetic Wilson lines in the Cartan subalgebra of  $SU(N)$ . Thus, when the  $N$  points are distinct, the DLCQ description is in a Coulomb phase where the low-energy gauge group is  $U(1)^{N-1}$ .

As above, the interactions of the model are determined by the period matrix  $\tau_{IJ} = \partial^2 \mathcal{F} / \partial a^I \partial a^J$  of the complex curve  $\Sigma$ . The presence of Wilson lines provides a natural weak-coupling expansion for the  $\sigma$ -model. It is convenient to choose the light-like Wilson lines so that  $\tau_j = (Z_{j+1} - Z_j) / 2\pi i = \alpha_j \tau_{\text{cl}}$ , for  $j = 1, 2, \dots, N$ , where the variables  $\{\alpha_j\}$  provide a partition of unity,

$$0 \leq \alpha_j \leq 1 \quad \sum_{j=1}^N \alpha_j = 1.$$

To obtain a weak-coupling limit we take  $\text{Im}\tau_{\text{cl}} \gg 1$  with the parameters  $\{\alpha_j\}$  held fixed. In this limit we may replace the prepotential by its classical value (2.11) and the target space  $\mathcal{M}$  reduces to,

$$\mathcal{M}_{\text{cl}} = \mathbb{R}^2 \times T^2[\tau_{\text{cl}}] \times \prod_{j=1}^N \frac{\text{Sym}^K(\mathbb{R}^2 \times T^2[\alpha_j \tau_{\text{cl}}])}{\mathbb{R}^2 \times T^2[\alpha_j \tau_{\text{cl}}]}$$

with a flat metric. In terms of the weak-coupling coordinates  $\{a^{ib}\}$  introduced in the previous section we have,

$$\begin{aligned} p_- &= H \\ &= \sum_{j=1}^N \sum_{b=1}^K \frac{R_-}{\alpha_j} \left[ \Pi^{jb} \bar{\Pi}^{jb} + \frac{1}{4\pi^2 \ell^2} |Q_e^{jb} - \alpha_j \bar{\tau}_{\text{cl}} Q_m^{jb}|^2 \right] \end{aligned} \quad (4.2)$$

Here  $\Pi^{jb} = (\alpha_j/R_-) \dot{a}^{jb}$  is the “covariant” momentum conjugate to  $a^{jb}$ .  $Q_e^{jb}$  and  $Q_m^{jb}$  are the conserved fibre momenta in the new coordinates which are also subject to the constraints,

$$\frac{1}{N} \sum_{b=1}^K Q_e^{jb} = \bar{Q}_e \quad \frac{1}{N} \sum_{b=1}^K Q_m^{jb} = \bar{Q}_m$$

for  $j = 1, 2, \dots, N$ . In the context of DLCQ, these constraints imply the freezing out of the centre of mass for each species of parton, reflecting IR divergences coming from the logarithmic growth of the Coulomb potential in two spatial dimensions [3]. The  $2\pi$  periodicity of the fibre coordinates  $\{\theta_e^I, \theta_{m,I}\}$  implies that each component of  $Q_e$  and  $Q_m$  is quantised in integer units<sup>12</sup>.

To interpret this free limit of the model we start with the case of vanishing fibre momentum;  $Q_e^{jb} = Q_m^{jb} = 0$ . In this case we recognise (4.2) as the Hamiltonian for  $KN$  non-relativistic particles moving freely on  $\mathbb{R}^2$ . Each Yang-Mills instanton has split up into  $K$  constituents which we will call partons. There are  $K$  identical partons in each of  $N$  species. Each parton of the  $j$ 'th species has mass  $\alpha_j/R_-$ . To compare with the  $\mathcal{N} = 4$  theory, we recall that, in DLCQ, a free relativistic particle behaves like a non-relativistic particle of mass  $p_+$  moving in two non-compact dimensions transverse to the light-cone. In the presence of the light-like Wilson lines described above, the quantisation rule for the null momentum  $p_+$  carried by each particle is modified for particles charged under the Cartan subgroup of  $SU(N)$ . For a particle carrying unit charge under the subgroup  $U(1)_j$  corresponding to the simple root<sup>13</sup>

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<sup>12</sup>We set  $\hbar = 1$  throughout.

<sup>13</sup>Here  $e_j$  denotes a matrix whose only non-zero entry is unity in the  $j$ 'th position on the diagonal and we work with the convention  $e_{N+1} = e_1$ .



$e_{j+1} - e_j$  of  $SU(N)$ , the quantisation rule is,

$$p_+ = \frac{\alpha_j + K_j}{R_-} \quad K_j \in \mathbb{N}.$$

Thus it is consistent for a single unit of  $p_+$  to be shared between  $N$  particles where the  $j$ 'th particle carries unit charge under  $U(1)_j$ . Putting these observations together, we see that (4.2) describes the dynamics associated with  $K$  units of  $p_+$ , each shared between  $N$  partons in this way. In other words, the model correctly reproduces the DLCQ of the free  $\mathcal{N} = 4$  theory.

To complete the identification with the free  $\mathcal{N} = 4$  theory we note that, in the full  $\sigma$ -model, the bosonic coordinates  $a, z$  of each parton are accompanied by fermions  $\psi^A$  and  $\bar{\psi}^{\bar{A}}$  in the  $\mathbf{4}$  and  $\bar{\mathbf{4}}$  of the enhanced  $SU(4)$  R-symmetry of the zero momentum sector. The fermions have canonical anti-commutation relations  $\{\psi^A, \bar{\psi}^{\bar{A}}\} \sim \delta^{A\bar{B}}$  and give rise to a fermionic Fock space in the usual way. Choosing a vacuum state  $|0\rangle$  annihilated by all the  $\bar{\psi}^{\bar{A}}$ , we find a tower of states associated with each parton,

$$|0\rangle, \quad \psi^A|0\rangle, \quad \psi^A\psi^B|0\rangle, \quad \psi^A\psi^B\psi^C|0\rangle, \quad \psi^A\psi^B\psi^C\psi^D|0\rangle$$

in the  $\mathbf{1} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \bar{\mathbf{4}} \oplus \mathbf{1}$  of  $SU(4)$ . These states fill out the usual on-shell light-cone supermultiplet of the  $\mathcal{N} = 4$  theory consisting of six scalars of helicity zero, four fermion components each of helicity  $\pm 1/2$  and the two transverse polarisations of the photon with helicity  $\pm 1$ . With a suitable charge assignment for the vacuum  $|0\rangle$ , the helicity corresponds to the  $U(1)$  generator  $\mathcal{R}$  of the  $\mathfrak{su}(1, 1|4)$  light-cone superconformal algebra given in (3.8) above.

Modes with non-zero values of the fibre momenta  $Q_e$  and/or  $Q_m$  have light-cone energy  $p_-$  which scales like  $1/\ell^2$ . They are naturally associated with the Kaluza-Klein modes of the  $(2, 0)$ -theory carrying momenta on the compactification torus. From the point of view of the low-energy gauge theory, modes with  $Q_e \neq 0$  are electrically charged and remain light at weak coupling. In contrast, modes with  $Q_m \neq 0$  are magnetically charged. Their masses are non-perturbative in  $g^2$  and they are very massive at weak coupling. In fact, each parton of the  $\mathcal{N} = 4$  theory has two infinite towers of Kaluza-Klein excitations which suggests a description in terms of free particles moving on the transverse  $\mathbb{R}^2 \times T^2$  of the full  $(2, 0)$

compactification. More precisely, if we focus on electrically charged states only, we can bring the Hamiltonian to the form,

$$\begin{aligned} p_- &= H \\ &= \sum_{j=1}^N \sum_{b=1}^K \frac{R_-}{\alpha_j} \left[ \Pi^{jb} \bar{\Pi}^{jb} + (P_e^{jb})^2 \right] \end{aligned} \quad (4.3)$$

corresponding to  $K$  free partons of mass  $\alpha_i/R_-$ , for each  $i = 1, 2, \dots, N$  moving on  $\mathbb{R}^2 \times S^1$  where the rescaled  $S^1$  has radius  $\ell$  and  $P_e^{jb} = Q_e^{jb}/2\pi\ell$  is the corresponding conserved momentum. This is consistent with the DLCQ description of five-dimensional free  $SU(N)$  gauge theory compactified on  $S^1$ .

Now we turn our attention to the interacting theory described by the full Hamiltonian (4.1). One general feature is the presence of  $2(KN - N + 1)$  conserved charges corresponding to the components of  $Q_e$  and  $Q_m$ . These correspond to the Kaluza-Klein momenta carried by the *individual* partons, which are obviously conserved in a limit where the theory becomes free. The surprising feature here is that they remain conserved in the interacting theory. Of course this is true only to the extent that the physics is correctly described by the semi-flat metric. As discussed above, interactions coming from one-loop and 4d instanton contributions to the prepotential lead to singularities in the semi-flat metric. To describe states with wavefunctions supported near these points, we need to resolve the singularities by including 3d instanton effects as in [14]. These however violate the isometries of the semi-flat metric and lead to non-conservation of the charges  $Q_e$  and  $Q_m$ .

At this point the question arises: what can we hope to calculate in this model? The spectrum of the light-cone Hamiltonian  $p_-$  is continuous and is hard to interpret, especially in an interacting conformal field theory. The dilatation operator, which is defined in the sector of zero fibre momentum, should have a discrete spectrum but the corresponding wavefunctions are centred near the origin which is highly singular. The results of the previous section suggest an answer: the model can be effectively regulated by introducing a worldline magnetic field proportional to the holomorphic symplectic form  $\eta$  on  $\mathcal{M}$ . This corresponds to changing the light-cone Hamiltonian  $H$  in (4.1) by the replacement  $P^I \rightarrow P^I - \kappa a^I$  for some parameter  $\kappa$  with the dimensions of mass. In the next section we will suggest a spacetime

interpretation for this modification of the DLCQ Hamiltonian. We define,

$$\mathbb{H} = \frac{1}{\mu} H \Big|_{P^I \rightarrow P^I - \kappa a^I}$$

where  $\kappa$  and  $\mu$  are two arbitrary mass scales. If we choose  $\kappa = \ell\mu/2R_-$  then, for  $P^I = 0$ ,  $\mathbb{H}$  reduces to,

$$\begin{aligned} \mathcal{L}_0[\mu] &= \frac{1}{\mu} H + \mu K \\ &= \frac{R_-}{\mu} (\text{Im } \tau^{-1})^{IJ} \Pi_I \bar{\Pi}_J + \frac{\mu}{R_-} \text{Im } \tau_{IJ} a^I \bar{a}^J + \text{fermions} \end{aligned}$$

which is isospectral to the dilatation operator of  $SU(1, 1|4)$  for any choice of the scale  $\mu$ . The eigenstates of this operator are localised near the origin and, for these states, the problem of singularities remains.

After a further rescaling  $a^I \rightarrow a^I/\kappa$ , to pass to dimensionless coordinates, the new Hamiltonian is given in full as<sup>14</sup>,

$$\begin{aligned} \mathbb{H} &= \frac{1}{\rho} \left[ (\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}} + \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}_A^{\bar{J}} \psi^{KB} \bar{\psi}_B^{\bar{L}} \right. \\ &\quad \left. + \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) \right] \\ &\quad + \rho \text{Im } \tau_{I\bar{J}} (P^I - a^I) (\bar{P}^{\bar{J}} - \bar{a}^{\bar{J}}) + \frac{1}{2} \text{Re} \left( \mathcal{F}_{I\bar{J}K}^{(3)} \Omega_{AB} \psi^{JA} \psi^{KB} (P^I - a^I) \right) \end{aligned}$$

where  $\rho = 4R_-/\mu\ell^2$ . The above relations tell us that states with generic non-zero values of the fibre momenta  $P^I$  are exponentially localised near the (generically) non-singular point  $a^I = P^I$ . Moreover, the localisation of the wavefunction is controlled by the dimensionless parameter  $\rho$ . It is particularly interesting to consider the limit  $\rho \rightarrow \infty$ , where the localisation is absolute and the relation  $a^I = P^I$  is imposed as a constraint on the phase space for all states of finite energy. As a consequence of the constraint, the coordinates  $a^I$  and  $z_I$  effectively become canonically conjugate variables. The limit therefore reduces the full phase space of the  $\sigma$ -model to the phase space of the complex integrable system  $(\mathcal{M}, \mathcal{J}, \eta)$  described in Section 2. In leading order semiclassical quantisation we consider wavefunctions of the form,

$$\Psi(z, \bar{z}) = \exp(i a^I z_I) \exp(i \bar{a}^{\bar{I}} \bar{z}_{\bar{I}}).$$

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<sup>14</sup>The shifting of the fermionic terms in the action follows from the homogeneity relation (2.7) for the prepotential  $\mathcal{F}(a)$ .

The Bohr-Sommerfeld condition corresponds to demanding that these wavefunctions are single-valued on the fibre  $\mathcal{J}(\Sigma)$ . Thus we find quantisation conditions,

$$a^I + \bar{a}^I \in \mathbb{Z} \qquad \tau_{IJ} a^J + \bar{\tau}_{IJ} \bar{a}^J \in \mathbb{Z}$$

or, more simply<sup>15</sup>

$$2\text{Re}[a^I] \in \mathbb{Z} \qquad 2\text{Re}[a_I^D] \in \mathbb{Z}$$

for  $I = 1, 2, \dots, r = KN - N + 1$ . The resulting spectrum is discrete and is also invariant under the  $Sp(2r, \mathbb{Z})$  monodromy group of the periods.

In the general case, the semiclassical quantisation conditions given above look unfamiliar. In particular, they differ from any of the quantisations of the same classical integrable system discussed in [25] which involve the quantisation of *either* the A-periods of the meromorphic differential  $\lambda$  *or* the B-periods (but not both). However, at least in the free limit of the  $\mathcal{N} = 4$  theory  $\tau \rightarrow i\infty$  (with arbitrary fixed light-like Wilson lines), we can make contact with a known quantum integrable system. In this limit, the classical system becomes an  $SL(N, \mathbb{C})$  Heisenberg spin chain. The semiclassical quantisation conditions described above coincide exactly with the semiclassical limit of a *quantum* integrable  $SL(N, \mathbb{C})$  spin chain with a principal series representation at each site. In particular, the  $N = 2$  case is worked out in detail in [26].

In the Lagrangian formulation, the same limit leads to a first-order Lagrangian of the form,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \eta_{uv} w^u \dot{w}^v + \frac{1}{2} \bar{\eta}_{\bar{u}\bar{v}} \bar{w}^{\bar{u}} \dot{\bar{w}}^{\bar{v}} \\ &= a^I \dot{z}_I + \bar{a}^I \dot{\bar{z}}_I \end{aligned}$$

where the first line gives the Lagrangian in manifestly covariant form where  $w_u, \bar{w}_{\bar{u}}$ , with  $u, \bar{u} = 1, 2, \dots, 2r$ , are generic holomorphic coordinates on  $\mathcal{M}$  for complex structure  $J$ . The second equality holds up to surface terms.

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<sup>15</sup> Note that for a scale-invariant prepotential, the homogeneity relation (2.7) implies  $a_I^D = \partial \mathcal{F} / \partial a^I = \tau_{IJ} a^J$ .

Finally we note that the effect of the  $\rho \rightarrow \infty$  limit is similar to that of the familiar limit which isolates the lowest Landau level for a particle in a background magnetic field [27]. This can be made explicit in the weak-coupling limit discussed above. Before the introducing the deformation, the electrically charged DLCQ partons (with  $Q_m = 0$ ) are governed by the Hamiltonian (4.3) for free motion on  $\mathbb{R}^2 \times S^1$ . The new Hamiltonian has the form,

$$\begin{aligned} p_- &= H \\ &= \sum_{j=1}^N \sum_{b=1}^K \frac{R_-}{\alpha_j} \left[ \Pi^{jb} \bar{\Pi}^{jb} + |P_e^{jb} - \alpha_j \tilde{\kappa} a^{jb}|^2 \right] \end{aligned} \quad (4.4)$$

where, as above,  $P_e^{jb} = Q_e^{jb}/2\pi\ell$  is the conserved momentum around the electric cycle of  $T^2$  and  $\tilde{\kappa}$  is a constant. Thus we see that the partons are moving in the presence of a background gauge field, where the light-cone momentum fraction  $\alpha_i$  plays the role of electric charge.

## 5 Spacetime interpretation

In the preceding sections we have chosen to modify the  $\sigma$ -model arising in the DLCQ description of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory by introducing a target space magnetic field. In this section we will propose a spacetime interpretation of this modification using the ideas of [28, 29].

We start from the  $\mathcal{N} = 4$  theory defined on Minkowski space  $\mathbb{R}^{1,3}$  with light-cone coordinates  $x_{\pm} = x_0 \pm x_1$  and transverse directions parametrised by a complex coordinate  $u = x_2 + ix_3$ . The spacetime metric is,

$$ds^2 = dx_+ dx_- + du d\bar{u}.$$

As above we choose a longitudinal null direction  $x_-$  which is compactified on a circle of radius  $R_-$  in DLCQ. The coordinate  $x_+$  is interpreted as ‘time’ and the light-cone Hamiltonian corresponds to the conjugate momentum  $p_-$ . A free massless particle in Minkowski space obeys the mass-shell condition  $p_+ p_- - \Pi_u \bar{\Pi}_u = 0$ , where  $\Pi_u$  is the conjugate momentum to  $u$ , which gives,

$$H = p_- = \frac{\Pi_u \bar{\Pi}_u}{p_+}.$$

This is the Hamiltonian for a non-relativistic particle of mass  $p_+$ . In a conformal theory the Hamiltonian is part of an  $SO(2, 1)$  group of conformal transformations which also includes a light-cone dilatation operator  $T = D + M_{01}$  and special conformal transformation  $K = K_0 + K_1$  with commutation relations,

$$[T, K] = 2iK \quad [T, H] = -2iH \quad [H, K] = -4iT.$$

For a free massless particle the additional generators are  $T = u\Pi_u + \bar{u}\bar{\Pi}_u$  and  $K = p_+u\bar{u}$ .

Minkowski space is conformally equivalent to the four-dimensional pp-wave geometry with metric,

$$ds^2 = dx_+dx_- + dud\bar{u} - \mu^2|u|^2dx_+^2 \quad (5.1)$$

where  $\mu$  is an arbitrary mass scale. The pp-wave geometry has null isometries corresponding to shifts of  $x_+$  and  $x_-$ . As before we take  $x_+$  as light-cone time and the conjugate momentum  $p_-$  as the Hamiltonian.

The conformal map which relates the pp-wave to Minkowski space maps the conformal generators of one space onto the other. According to [28, 29] we have,

$$H_{\text{pp-wave}} = H + \mu^2 K. \quad (5.2)$$

Up to an overall factor of  $\mu$ , it follows that  $H_{\text{pp-wave}}$  has the same spectrum as the Minkowski space light-cone dilatation operator  $T$ . The conformal equivalence means that, up to global issues, the  $\mathcal{N} = 4$  theory on the pp-wave is equivalent to the theory on Minkowski space with the appropriate identification of conformal generators. In particular, to determine the spectrum of the light-cone dilatation operator of the  $\mathcal{N} = 4$  theory in Minkowski space, one can equivalently study the light-cone Hamiltonian of the  $\mathcal{N} = 4$  theory in the plane wave geometry (5.1)<sup>16</sup>.

To construct the  $\mathcal{N} = 4$  theory on a pp-wave geometry one can hope to start from the  $(2, 0)$  theory on a six-dimensional space which reduces at large distances to a four-dimensional

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<sup>16</sup>This statement can be compared with the more familiar equivalence of the dilatation operator on  $\mathbb{R}^{1,3}$  and the Hamiltonian of the same theory on  $\mathbb{R} \times S^3$  provided by radial quantisation.

pp-wave after compactification on a torus. To realise this idea, we start by considering six-dimensional Minkowski space in light-cone coordinates  $x_{\pm} = x_0 \pm x_1$  where the transverse space is parametrised by complex coordinates  $u = x_2 + ix_3$  and  $v = x_4 - ix_5$ . The Minkowski space metric is

$$ds^2 = dx_+ dx_- + du d\bar{u} + dv d\bar{v}.$$

To obtain a four-dimensional theory we compactify  $x_4$  and  $x_5$  on circles of radius  $R_4$  and  $R_5$  respectively, giving the identifications,

$$v \sim v + 2\pi R_4 \quad v \sim v + 2\pi i R_5$$

To obtain  $\mathcal{N} = 4$  SUSY Yang-Mills at low energies with coupling  $g^2$  (and  $\theta = 0$ ) we set  $R_5/R_4 = \text{Im}\tau_{\text{cl}} = 4\pi/g^2$ . In light-cone coordinates the light-cone Hamiltonian for a free particle in six-dimensions takes the form,

$$H = p_- = \frac{1}{p_+} [\Pi_u \bar{\Pi}_u + \Pi_v \bar{\Pi}_v]$$

where  $\Pi_u$  and  $\Pi_v$  are conjugate momenta to the complex coordinates  $u$  and  $v$  respectively. Thus, the free particle in six dimensions becomes a free non-relativistic particle of mass  $p_+$  moving on the transverse space  $\mathbb{R}^2 \times T^2$ . This is the starting point for a DLCQ description of a six-dimensional QFT compactified to four dimensions on  $T^2$ .

In order to make contact with the ideas of the previous sections we consider instead a six-dimensional space where the light-like direction  $x_-$  is fibred non-trivially over the other dimensions with metric,

$$ds^2 = dx_+ (dx_- + \mu u dv + \mu \bar{u} d\bar{v}) + du d\bar{u} + dv d\bar{v} \quad (5.3)$$

where  $\mu$  is a mass scale. This is a Bargmann space of the type considered in [29]. The resulting light-cone Hamiltonian takes the form,

$$\tilde{H} = p_- = \frac{1}{p_+} \Pi_u \bar{\Pi}_u + \frac{1}{p_+} (\Pi_v - p_+ \mu u) (\bar{\Pi}_v - p_+ \mu \bar{u}). \quad (5.4)$$

Thus the effect of the fibration on a non-relativistic particle is to give it an electric charge equal to its mass  $p_+$  and couple it to a background anti-self-dual magnetic field ,

$$f = \mu (du \wedge dv + d\bar{u} \wedge d\bar{v}). \quad (5.5)$$

This matches the weak coupling limit (4.4) of the DLCQ Hamiltonian. Thus, the deformation of the  $\sigma$ -model obtained by introducing target space magnetic field is a natural candidate for the DLCQ description of the  $(2, 0)$  theory compactified on the Bargmann space (5.3).

In the case that the fibre momentum vanishes, the Hamiltonian reduces to,

$$H_{\text{pp-wave}} = \frac{1}{p_+} \Pi_u \bar{\Pi}_u + \mu^2 p_+ u \bar{u}.$$

This can also be understood by completing the square to put the six-dimensional metric (5.3) in the form

$$ds^2 = dx_+ dx_- + dud\bar{u} - \mu^2 |u|^2 dx_+^2 + (dv - \bar{u} dx_+) (d\bar{v} - u dx_+) \quad (5.6)$$

which manifestly has the form of a  $T^2$  bundle over the four-dimensional pp-wave geometry (5.1).

We will support this interpretation further by analysing the effect of the anti-self-dual magnetic field (5.5), whose potential is

$$a = \mu (udv + \bar{u}d\bar{v}),$$

in the DLCQ description of the  $(2, 0)$  theory compactified on  $\mathbb{R}^2 \times T^2$ . Thus we investigate the effect of this background field on slowly-moving Yang-Mills instantons. Since the effective electric charge  $p_+$  in (5.4) is set by the instanton number  $p_+ = K/R_-$ , the corresponding gauge potential  $a$  must couple minimally to the instanton current,

$$j = \frac{1}{8\pi^2} * \text{tr } F \wedge F.$$

The resulting five-dimensional action is the  $\mathcal{N} = 2$  supersymmetric completion of

$$S = -\frac{1}{g^2} \int_{\mathbb{R}^{2,1} \times T^2} \text{tr} (F \wedge *F + F \wedge F \wedge a), \quad (5.7)$$

where  $g^2 = R_-/8\pi^2$  and the normalisation is chosen so that a  $K$ -instanton has electric charge  $p_+ = K/R_-$ . From the stringy perspective this can be understood as a modification of the D4-brane action due to a background Ramond-Ramond 1-form flux proportional to  $a$ .



A surprising fact is that, by virtue of the anti-self-duality of  $f$ , a static, self-dual instanton is still an exact solution of the modified equations of motion

$$D_\nu F^{\nu\mu} - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\tau} F_{\nu\rho} f_{\sigma\tau} = 0.$$

It therefore makes sense to work with the moduli space approximation. The standard procedure is to promote the collective coordinates  $X^\alpha$ ,  $\alpha = 1, \dots, 4(KN - N + 1)$ , to slowly varying dynamical degrees of freedom and set

$$A_0 = \dot{X}^\alpha \Omega_\alpha, \quad A_m = A_m^{inst}(x; X^\alpha(t)) \quad m = 1, 2, 3, 4$$

where  $\Omega_\alpha$  is the gauge-fixing parameter defined by

$$D_m \delta_\alpha A_m := D_m (\partial_\alpha A_m - D_m \Omega_\alpha) = 0.$$

One finds that the equations of motion are solved to lowest order in time derivatives. Substituting this ansatz into the action (5.7) leads to

$$S = \frac{1}{g^2} \int dt - \frac{1}{2} \mathcal{G}_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta + \mathcal{A}_\alpha \dot{X}^\alpha,$$

where

$$\mathcal{G}_{\alpha\beta} = -2 \int d^4x \operatorname{tr} (\delta_\alpha A_m \delta_\beta A_m)$$

is the usual metric on moduli space and we interpret

$$\mathcal{A}_\alpha = -2 \int d^4x \operatorname{tr} (\delta_\alpha A_m F_{mn} a_n)$$

as the instanton worldline magnetic field induced by  $a$ .

The field strength associated to  $\mathcal{A}$  is

$$\mathcal{F}_{\alpha\beta} = -2 \int d^4x \operatorname{tr} (\delta_\alpha A_m f_{mn} \delta_\beta A_n).$$

But the holomorphic symplectic form on  $\mathbb{R}^2 \times T^2$  (with respect to the preferred complex structure induced by holomorphic coordinates  $u$  and  $v$ ) is just  $du \wedge dv$ . By the usual rules for inducing the hyper-Kähler structure on moduli space from that of Euclidean space, the corresponding holomorphic symplectic form  $\eta$  on moduli space is given in terms of  $f$  and  $\mathcal{F}$  by

$$\mathcal{F}_{\alpha\beta} = \mu (\eta + \bar{\eta})_{\alpha\beta} = -2 \int d^4x \operatorname{tr} (\delta_\alpha A_m f_{mn} \delta_\beta A_n).$$

We conclude that the effect of the spacetime magnetic field arising from Kaluza-Klein reduction of the Bargmann geometry (5.3), is to produce exactly the deformation used in section 3 to regularise the DLCQ description of the  $\mathcal{N} = 4$  theory.

## 6 Discussion

In this paper we have presented further evidence in favour of the DLCQ description of the  $\mathcal{N} = 4$  theory proposed in [3, 4]. The most important question however is whether and to what extent the model can be used to calculate gauge theory observables. An obvious goal is to reproduce the planar anomalous dimensions of the  $\mathcal{N} = 4$  theory, hopefully shedding new light on the underlying integrable structure. Although our results are preliminary, they suggest a possible approach to this problem. The idea is to work with the deformed model in the  $\rho \rightarrow \infty$  limit where we have already established semiclassical integrability. A full treatment would also include the fluctuations of both bosonic and fermionic fields around the leading order solution discussed above. Next one seeks a quantisation of this system which preserves the semiclassical integrability. The weak coupling limit described above where the system can be related to a known spin chain looks particularly promising in this regard. Interestingly, in addition to a naïve weak coupling limit  $\tau \rightarrow i\infty$ , the connection to a spin chain seems to hold also in a 't Hooft limit where  $N$  is also taken to infinity holding  $N/\text{Im}\tau$  fixed.

The next step is to focus on states in the sector with  $Q_e = Q_m = 0$  corresponding to the gauge-invariant states of the  $\mathcal{N} = 4$  theory. In fact the integrable system arising *after* taking the  $\rho \rightarrow \infty$  limit seems to extend smoothly to this sector despite the fact that the model is singular for states with  $Q_e = Q_m = 0$  for any finite value of  $\rho$ . Of course, if this approach is correct, we would expect to find some obstruction to integrability away from the large- $N$  't Hooft limit. We would also expect to find some relation between the quantum corrected eigenvalues of the dilatation operator and the conserved charges of the integrable system. The final step is to take a  $K \rightarrow \infty$  limit to decompactify the light-like circle and compare with the  $\mathcal{N} = 4$  theory in Minkowski space. These issues are currently under investigation [7].

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## References

- [1] N. Beisert, C. Ahn, L. Alday, Z. Bajnok, J. Drummond, et al., *Review of AdS/CFT Integrability: An Overview*, *Lett.Math.Phys.* **99** (2012) 3–32, [[arXiv:1012.3982](#)].
- [2] F. Delduc, M. Magro, and B. Vicedo, *Alleviating the non-ultralocality of the  $AdS_5 \times S^5$  superstring*, *JHEP* **1210** (2012) 061, [[arXiv:1206.6050](#)].
- [3] O. Ganor and S. Sethi, *New perspectives on Yang-Mills theories with sixteen supersymmetries*, *JHEP* **9801** (1998) 007, [[hep-th/9712071](#)].
- [4] A. Kapustin and S. Sethi, *The Higgs branch of impurity theories*, *Adv.Theor.Math.Phys.* **2** (1998) 571–591, [[hep-th/9804027](#)].
- [5] E. Witten, *Some comments on string dynamics*, [hep-th/9507121](#).
- [6] N. Dorey and A. Singleton, *Superconformal Quantum Mechanics and the Discrete Light-Cone Quantisation of  $N=4$  SUSY Yang-Mills*, [arXiv:1409.8440](#).
- [7] N. Dorey, *Work in progress*, .
- [8] N. Nekrasov, *Holomorphic bundles and many body systems*, *Commun.Math.Phys.* **180** (1996) 587–604, [[hep-th/9503157](#)].
- [9] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three-dimensions*, [hep-th/9607163](#).
- [10] A. Kapustin, *Solution of  $N=2$  gauge theories via compactification to three-dimensions*, *Nucl.Phys.* **B534** (1998) 531–545, [[hep-th/9804069](#)].
- [11] N. Dorey, T. Hollowood, and S. Kumar, *An Exact elliptic superpotential for  $N=1^*$  deformations of finite  $N=2$  gauge theories*, *Nucl.Phys.* **B624** (2002) 95–145, [[hep-th/0108221](#)].
- [12] K. Intriligator and N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, *Phys.Lett.* **B387** (1996) 513–519, [[hep-th/9607207](#)].
- [13] E. Witten, *Solutions of four-dimensional field theories via M theory*, *Nucl.Phys.* **B500** (1997) 3–42, [[hep-th/9703166](#)].

- [14] D. Gaiotto, G. Moore, and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Commun.Math.Phys.* **299** (2010) 163–224, [[arXiv:0807.4723](#)].
- [15] K.-M. Lee and P. Yi, *Monopoles and instantons on partially compactified D-branes*, *Phys.Rev.* **D56** (1997) 3711–3717, [[hep-th/9702107](#)].
- [16] M. Luty and W. Taylor, *Varieties of vacua in classical supersymmetric gauge theories*, *Phys.Rev.* **D53** (1996) 3399–3405, [[hep-th/9506098](#)].
- [17] I. Krichever, O. Babelon, E. Billey, and M. Talon, *Spin generalization of the Calogero-Moser system and the matrix KP equation*, [hep-th/9411160](#).
- [18] A. Gorsky, A. Marshakov, A. Mironov, and A. Morozov,  *$N=2$  supersymmetric QCD and integrable spin chains: Rational case  $N(f) < 2N(c)$* , *Phys.Lett.* **B380** (1996) 75–80, [[hep-th/9603140](#)].
- [19] A. Gorsky, S. Gukov, and A. Mironov, *Multiscale  $N=2$  SUSY field theories, integrable systems and their stringy / brane origin. 1.*, *Nucl.Phys.* **B517** (1998) 409–461, [[hep-th/9707120](#)].
- [20] L. Alvarez-Gaumé and D. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, *Commun.Math.Phys.* **80** (1981) 443.
- [21] J. Figueroa-O’Farrill, C. Köhl, and B. Spence, *Supersymmetry and the cohomology of (hyper)Kähler manifolds*, *Nucl.Phys. B* **503** (1997) 614–626, [[hep-th/9705161](#)].
- [22] M. Verbitsky, *Action of the Lie algebra  $SO(5)$  on the cohomology of a hyperkähler manifold*, *Functional Analysis and Its Applications* **24** (1990), no. 3 229–230.
- [23] Y. Nishida and D. Son, *Nonrelativistic conformal field theories*, *Phys.Rev.* **D76** (2007) 086004, [[arXiv:0706.3746](#)].
- [24] H.-C. Kim, S. Kim, E. Koh, K. Lee, and S. Lee, *On instantons as Kaluza-Klein modes of M5-branes*, *JHEP* **1112** (2011) 031, [[arXiv:1110.2175](#)].
- [25] N. Nekrasov and S. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, [arXiv:0908.4052](#).

- [26] S. Derkachov, G. Korchemsky, and A. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD. 3. Quasiclassical approach*, *Nucl.Phys.* **B661** (2003) 533–576, [[hep-th/0212169](#)].
- [27] L. Landau and E. Lifshitz, *Quantum Mechanics (Non-relativistic Theory)*. Butterworth-Heinmann, 1997.
- [28] J. Maldacena, D. Martelli, and Y. Tachikawa, *Comments on string theory backgrounds with non-relativistic conformal symmetry*, *JHEP* **0810** (2008) 072, [[arXiv:0807.1100](#)].
- [29] C. Duval, P. Horvathy, and L. Palla, *Conformal properties of Chern-Simons vortices in external fields*, *Phys.Rev.* **D50** (1994) 6658–6661, [[hep-th/9404047](#), [hep-th/9405229](#)].